STOCHASTIC RESONANCE IN COUPLED THRESHOLD ELEMENTS WITH INPUT SIGNALS SHIFTED IN PHASE

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Stochastic resonance in a system of two coupled threshold elements (neurons) forming a small artificial neural network is investigated. The elements have either antisymmetric or logistic (binary) response function and are driven by periodic signals and independent noises. Periodic signals at their inputs have equal amplitudes and frequencies but are shifted in phase. Depending on the response function and the phase shift, enhancement of stochastic resonance in individual elements, characterized by the output signal-to-noise ratio, and stochastic resonance with a spatiotemporal input signal, characterized by the correlation function between the input and output signals, are observed for proper coupling between elements.

1. Introduction

In certain, mainly nonlinear systems driven by a combination of a periodic signal and noise a phenomenon of stochastic resonance (SR) occurs. The essence of this phenomenon is that intensity of a periodic component of a properly defined output signal can be maximized against the output noise intensity for nonzero input noise intensity. The most widely investigated models of SR are based upon bistable dynamical systems, threshold-crossing detectors and biological neuron models. For a few years, SR has also been investigated in high-dimensional systems. In this case ensembles of many coupled elements exhibiting SR are most often studied. Such models include e.g. arrays of globally or locally coupled bistable and threshold elements, networks of model biological neurons, excitable systems, spatially extended systems, the Ising model etc.

In coupled systems usually the case when individual elements are driven by a common periodic signal and independent uncorrelated noises is considered. SR in the output signal of one element or of a whole system is then investigated and compared to SR in a single uncoupled element. It is
known that proper coupling enhances SR. For example in chains of diffusively coupled stochastic bistable elements\textsuperscript{27–29} there exists an optimum coupling strength and optimum noise for which SR in a single element is significantly enhanced over SR in an uncoupled element. The enhancement of SR appears as a result of spatiotemporal synchronization of all elements to the input signal and among themselves, and this effect is called array enhanced SR. The case when individual elements are driven by different signals is less frequently investigated. For example in Ref. 24 a neuronal network was considered with bias and signal strength distributed on the network. In Ref. 35 a one-dimensional Ising chain was investigated in which the input signal (the magnetic field) was periodic in space, but constant in time, and the thermal fluctuations played a role of internal noise. Thus it seems interesting to study models in which different elements are driven by different signals, and to check if also in this case the enhancement of SR due to coupling occurs.

In this paper we consider two threshold elements (formal neurons) coupled in a small artificial neural network (ANN). Such elements are known to exhibit SR\textsuperscript{6–10} and can be used for qualitative simulations of SR in biological neuron models.\textsuperscript{7} Here, the elements are driven by subthreshold periodic signals which have the same amplitude and frequency, but different phase at the input of each neuron, and by independent noises. In spatially extended systems this can be a typical situation since signals in two distant points can be shifted in phase due to a finite propagation time of the signal. From a different point of view, the small ANN is driven by a spatiotemporal periodic signal which consists of two periodic signals shifted in phase acting on two spatially separate threshold elements. We are interested both in the possible enhancement of SR in an individual element due to coupling and in the maximization of the correlation between the outputs of the two elements and the spatiotemporal periodic input signal as the noise intensity and coupling are varied. The latter quantity is a measure of another kind of SR for the whole ANN which we call SR with a spatiotemporal periodic input signal.

Our investigations are based on numerical simulations and on simple theoretical considerations. The preliminary findings have been partly reported in our previous paper.\textsuperscript{36} Here, the considerations are extended to a much larger class of models. Moreover, analysis of correlation with the spatiotemporal periodic input signal is performed and carefully discussed.

2. The Systems and Methods of Analysis

We investigate two coupled threshold elements (neurons) denoted as $i = 1, 2$ forming a small ANN with parallel updating. The time in our model, as usually in ANN, is discrete rather than continuous, i.e. the state of the network is updated in consecutive iterations $n = 0, 1, 2, \ldots$. Both elements are driven by periodic signals $s_n^{(i)}$, with amplitude $A$ and frequency $\omega$, and by independent white Gaussian noises $\eta_n^{(i)}$ with variance $D$. The initial phase of $s_n^{(1)}$ is $\phi$, while $s_n^{(2)}$ is shifted in phase...
by $\Delta \phi$. These two signals together form the spatiotemporal periodic input signal acting on two spatially separate threshold elements. The coupling strength (the synaptic connection weight) $w$ is symmetric; as usually, $w > 0$ is called excitatory and $w < 0$ — inhibitory coupling. We consider typical formal neurons, i.e. two-state threshold elements with threshold $b$ and either with antisymmetric or logistic (binary) response function. By the antisymmetric response function we understand $f(z) = 2[\Theta(z) - 0.5]$, where $\Theta$ is the Heaviside step function, while the logistic (binary) response function is $f(z) = \Theta(z)$. The output of the element $i$ at time $n$ is $x_n^{(i)} = f(X_{n-1}^{(i)} - b)$, where $X_n^{(i)}$ is the total input to the element $i$ at time $n$. Thus if $X_{n-1}^{(i)} < b$ then $x_n^{(i)} = -1$ or $x_n^{(i)} = 0$ in the case of antisymmetric and logistic response function, respectively (quiescent state). If $X_{n-1}^{(i)} > b$ then $x_n^{(i)} = 1$ for both kinds of the response function (firing). The equations for the time dependence of $x_n^{(i)}$ are

$$
\begin{align*}
  x_{n+1}^{(1)} &= f \left[ A \sin(\omega_n + \phi) + \eta_n^{(1)} + wx_n^{(2)} - b \right], \\
  x_{n+1}^{(2)} &= f \left[ A \sin(\omega_n + \phi + \Delta \phi) + \eta_n^{(2)} + wx_n^{(1)} - b \right].
\end{align*}
$$

(1)

It should be noted that systems (1) with the antisymmetric and logistic response functions are not equivalent to each other for given $w$, $b$, $\eta_n^{(1)}$ and $s_n^{(i)}$. Equivalence of a formal neuron with binary output to a neuron with $\pm 1$ output requires a transformation of both the coupling strength $w$ and the total input to the neuron.\(^{37}\)

In this paper we use two different measures to characterize SR: the signal-to-noise ratio (SNR) and the correlation function between the spatiotemporal periodic input signal and the outputs of the threshold elements $C$. SR in an individual element $i$ is characterized by the SNR\(^{(i)}\) which is obtained from the power spectrum density $S^{(i)}(\omega)$ of the respective time series $x_n^{(i)}$. We define $\text{SNR}^{(i)} = 10 \log[S_p^{(i)}(\omega_s)/S_N^{(i)}(\omega_s)]$, where $S_p^{(i)}(\omega_s) = S^{(i)}(\omega_s) - S_N^{(i)}(\omega_s)$ is the height of the peak in the power spectrum density at $\omega = \omega_s$, and $S_N^{(i)}(\omega_s)$ is the noise background in the vicinity of $\omega_s$. In systems with SR, the SNR as a function of $D$ has a maximum for some $D > 0$.\(^2\) SR in the whole ANN is characterized by the following correlation function

$$
C = \frac{1}{2} (C^{(1)} + C^{(2)}), \quad C^{(i)} = \frac{\langle x_n^{(i)} s_n^{(i)} \rangle}{\sqrt{\langle (s_n^{(i)})^2 \rangle \left[ \langle (x_n^{(i)})^2 \rangle - \langle x_n^{(i)} \rangle^2 \right]}}
$$

(2)

where the brackets denote the time average. $C$ is a typical normalized input–output correlation function obtained under the assumption that $\langle s^{(i)} \rangle = 0$ and averaged over the two elements. We show that also $C$ can have a maximum as $D$ is varied which means that for some noise intensity the spatiotemporal periodic signal is best processed by our system. Thus SR with a spatiotemporal periodic signal is found.
3. Numerical Results and Their Discussion

In this section results concerning SR in the system (1) based on numerical simulations are summarized. Only slow in time periodic input signals with $\omega_s \rightarrow 0$ are considered; typically the period $T_s = \omega_s/2\pi = 128$ was used. Some results on SR in Eq. (1) with fast input signals can be found in our previous work.\textsuperscript{36} It is enough to limit the phase shift to $0 \leq \Delta \phi \leq \pi$. For such phase shifts the spatial variation of the spatiotemporal periodic input signal can be much faster than the temporal one. If $\Delta \phi = 0$ (both elements equivalent) or $\Delta \phi = \pi$ (one full spatial wavelength contained within the ANN) the system symmetry is retained while in other cases it is broken and the results for SR in different elements need not be identical.

In our numerical simulations we use the following parameters: $A = 0.5$, $b = 0.6$. The power spectrum densities are evaluated from $N = 4096$ points of the time series $x_n^{(i)}$, $i = 1, 2$, and averaged over 100 consecutive runs. Then, SNR$^{(i)}$ are evaluated and averaged over 10 random initial conditions $x_0^{(1)}$, $x_0^{(2)}$ and $\phi$. All results for the SNR are normalized to the frequency bandwidth $\Delta f = 2^{-12}$ Hz.\textsuperscript{3}

3.1. Threshold elements with the antisymmetric response function

Numerical results for the ANN which consists of threshold elements with the antisymmetric response function are shown in Fig. 1. In the left column SNR$^{(1)}$ and in the right one $C$ versus the noise variance $D$ are depicted, while the phase difference $\Delta \phi$ increases from 0 to $\pi$ in rows of Fig. 1 from top to bottom. Figures 1(a), (b) are obtained for $\Delta \phi = 0$. Figures 1(c), (d) are obtained for $\Delta \phi = \pi/4$ but the results are typical of $0 < \Delta \phi \leq \pi/2$. Figures 1(e), (f) are obtained for $\Delta \phi = \pi$ but the results are typical of $\pi/2 < \Delta \phi \leq \pi$. The curves for different connection strengths $w$ are labeled by numbers. The curves (2) were obtained with $w = -0.1$ which is close to the case of uncoupled elements, and in fact it was checked that they do not differ much from the curves obtained with $w = 0$.

First, SR in an individual element is discussed whose measure is the SNR. SR is observed, i.e. the curves SNR$^{(1)}$ versus $D$ show maxima, for all $\Delta \phi$ and in a wide range of $w$ except of moderate inhibitory coupling ($w = -1.0$ in Fig. 1(a), $w = -0.5$ and $w = -1.0$ in Figs. 1(c), (e)). If the maximum occurs it is most pronounced for almost uncoupled elements, e.g. for $w = -0.1$, and the quality of SR, i.e. the height of the maximum is deteriorated by any coupling independently of the phase shift. For moderate inhibitory coupling SR disappears and the curves SNR$^{(1)}$ versus $D$ are monotonically decreasing. In this case the maximum value of SNR$^{(1)}$ which occurs for $D = 0$ can (Figs. 1(c), (e), $\Delta \phi > 0$) but does not have to (Fig. 1(a), $\Delta \phi = 0$) exceed the maxima connected with the appearance of SR for other $w$. Hence processing a periodic signal by an individual element can be enhanced due to moderate inhibitory coupling to another element. This can happen if there is nonzero phase shift between the signals at inputs of the two elements. However, this effect is not connected with the enhancement of SR which disappears for such coupling, since the largest value of the SNR occurs for $D = 0$. 
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Fig. 1. Numerical results for threshold elements with the antisymmetric response function: (a, c, e) SNR\(^{(1)}\) versus \(D\); (b, d, f) correlation function \(C\) versus \(D\). Phase shifts (in units of 2\(\pi\)): (a, b) \(\Delta \phi = 0.0\), (c, d) \(\Delta \phi = 0.125\), (e, f) \(\Delta \phi = 0.5\). Numbers label curves obtained for different coupling: (1) \(w = 1.0\), (2) \(w = -0.1\), (3) \(w = -0.5\), (4) \(w = -1.0\), (5) \(w = -1.5\).

We now turn to the problem of SR with a spatiotemporal input signal whose measure is the correlation function \(C\). This kind of SR is observed for all \(\Delta \phi\) and in a similar range of \(w\) as SR in an individual element, i.e. the curves \(C\) versus \(D\) show maxima apart from the case of moderate inhibitory coupling. However, important differences between respective curves in the left and right columns of Fig. 1 can be seen. For example it is possible that the maximum value of \(C\) increases with proper coupling above that for almost uncoupled elements and the property of SR is retained, i.e. the maximum occurs for \(D > 0\); see e.g. curves for \(w = -0.1\), \(w = -1.0\) and \(w = -1.5\) in Fig. 1(f). On the other hand, for moderate inhibitory coupling, e.g. for \(w = -0.5\) in Figs. 1(b), (d), (f), \(C\) becomes a monotonically decreasing function of \(D\), SR disappears, and the maximum value of the correlation function
for $D = 0$ exceeds the maxima connected with the appearance of SR for other $w$ independently of the phase shift, even for $\Delta \phi = 0$. Hence it can be concluded that, in our system, SR with a spatiotemporal input signal occurs for all phase shifts $\Delta \phi$ and in a wide range of coupling strengths. Moreover, this effect can be enhanced due to coupling. This happens for large phase shifts, e.g. $\Delta \phi = \pi$, and quite large inhibitory coupling (Fig. 1(f)); in other cases SR is best pronounced for vanishing coupling (Figs. 1(b), (d)). However, for any phase shift $\Delta \phi$ the spatiotemporal input signal is best processed in the case when SR does not occur, i.e. for moderate inhibitory coupling and for zero noise.

It should be noted that in most cases the maxima of the curves $\text{SNR}^{(1)}$ versus $D$ and $C$ versus $D$ for a given $w$ do not coincide. This is because the SNR and the correlation function are quite different measures of SR which are sensitive to different properties of the output signal.

The results of Fig. 1 can be also qualitatively interpreted in terms of synchronization of firing of the two threshold elements. Excitory coupling increases the probability that if one element at a given iteration is in a given state then the other element at the next iteration will be in this state, too. In contrast, inhibitory coupling increases the probability that the other element will be in the opposite state. The best signal processing is expected when the two neurons fire in phase with the periodic signals at their inputs. If the phase shift between these signals is small this condition means that they should also fire in phase (synchrony) with each other. This happens most likely when $w > 0$. On the contrary, if the phase shift is large and close to $\pi$ the firing of the neurons should be also shifted in phase by c.a. $\pi$. This requires $w < 0$. The above considerations are confirmed by Fig. 1. It can be seen that in general the curves for $w > 0$ shift down and the curves for $w < 0$ shift up with the rise of $\Delta \phi$.

![Fig. 2. Numerical results for threshold elements with the antisymmetric response function: $	ext{SNR}^{(i)}$ versus $D$, $i = 1, 2$. For phase shift $\Delta \phi = 0.125$ (in units of $2\pi$) results for the neuron 1 (solid curves) and 2 (dots) are shown. Numbers label curves obtained for different coupling: (1) $w = 0.5$, (2) $w = -0.5$, (3) $w = -1.5$.](image)
Fig. 3. Numerical results for threshold elements with the logistic (binary) response function: (a, c, e) $\text{SNR}(1)$ versus $D$; (b, d, f) correlation function $C$ versus $D$. Phase shifts (in units of $2\pi$): (a, b) $\Delta\phi = 0$, (c, d) $\Delta\phi = 0.375$, (e, f) $\Delta\phi = 0.5$. Numbers label curves obtained for different coupling: (1) $w = 2.0$, (2) $w = 1.0$, (3) $w = 0.5$, (4) $w = -0.1$, (5) $w = -2.0$.

As pointed out above, due to the symmetry breaking when $\Delta\phi \neq 0$ and $\Delta\phi \neq \pi$ the two threshold elements need not behave in the same way. This is confirmed by Fig. 2 in which both $\text{SNR}(1)$ and $\text{SNR}(2)$ are shown for $\Delta\phi = \pi/4$. Although the curves for both elements are qualitatively similar the quantitative difference between them is significant, in particular for $w < 0$. This difference disappears for $\Delta\phi = 0$ and $\Delta\phi = \pi$.

3.2. Threshold elements with the logistic response function

Numerical results for the ANN which consists of threshold elements with the logistic response function are shown in Fig. 3. The results for the SNR in this case were discussed in our previous work.\textsuperscript{36} The way of presentation is similar as in the case of antisymmetric response function. Figures 3(a), (b) are obtained for $\Delta\phi = 0$ but the
results are typical of $0 \leq \Delta \phi \leq \pi/2$. Figures 3(c), (d) and Figs. 3(e), (f) are obtained for $\Delta \phi = 3\pi/4$ and $\pi$, respectively, but the results are typical of $\pi/2 < \Delta \phi \leq \pi$.

For any $\Delta \phi$ the curves SNR$^{(1)}$ versus $D$ and $C$ versus $D$ show maxima in a wide range of $w$, possibly apart from the case of large excitory coupling and large phase shifts where the curves are monotonically increasing within the range of $D$ investigated. Thus SR in individual elements and SR with a spatiotemporal input signal are observed for almost any coupling. Moreover, any of the kinds of SR never disappears, i.e. the curves do not become monotonically decreasing for any $w$. Thus the best signal processing either by individual elements or by the whole ANN is always connected with the occurrence of SR.

The most important finding is that both kinds of SR (connected with the SNR and $C$) are enhanced due to proper coupling for any phase shift $\Delta \phi$. For example in the case $\Delta \phi = 0$ (Figs. 3(a), (b)) there exists an optimum value of coupling $w \approx 1$ for which the maxima of the SNR and $C$ are most pronounced. Similar dependence of the SNR and the correlation function on the coupling strength is observed for all $0 \leq \Delta \phi \leq \pi/2$, and the optimum coupling is always $w \approx 1$. This means that for optimum coupling SR in an individual element is maximized and, for the same coupling, synchronization between the spatiotemporal periodic signal and the outputs of the threshold elements is also maximized. Thus SR in an individual element and SR with a spatiotemporal signal are related to each other. This situation resembles optimization of coupling so that to obtain the highest SNR in an individual bistable stochastic element in array enhanced SR. In array enhanced SR the largest enhancement of SR in an individual element appears as a result of spatiotemporal synchronization among all elements. In the present case this condition is replaced by the requirement of maximum synchronization of the whole ANN to the spatiotemporal input signal since the input signal itself is spatially nonuniform. However, this analogy with array enhanced SR cannot be extended too far. For array enhanced SR the maximum enhancement of SR and the best spatiotemporal synchronization for a given coupling occur for the same noise intensity, while in the present case the maxima of the curves SNR$^{(1)}$ versus $D$ and $C$ versus $D$ are shifted with respect to each other (cf. e.g. curves for $w = 1$ in Figs. 3(a) and (b)).

In the case $\pi/2 < \Delta \phi \leq \pi$ the enhancement of SR is not so obvious (Figs. 3(c)–(f)). The maxima of the curves SNR versus $D$ and $C$ versus $D$ decrease with excitory coupling and remain unchanged for inhibitory coupling. The only visible effect of $w < 0$ is shifting upwards the above-mentioned curves for large $D$. In this case there is no optimum value of coupling and the best enhancement of the SNR and $C$ for large $D$ occurs when $w \rightarrow -\infty$.

The results of Fig. 3 can also be qualitatively interpreted using arguments concerning the synchronization of firing of the two threshold elements, like in Sec. 3.1. It was also observed that in the limit $\omega_\omega \rightarrow 0$ there is no significant difference between SNR$^{(1)}$ and SNR$^{(2)}$ for any $\Delta \phi$ within the range of $D$ investigated.
4. Theory

In this section we extend a theory of SR in a threshold element with discrete time to the case of two coupled elements. The quantities SNR and C in threshold elements can be evaluated analytically provided the probability that \( x_n^{(i)} = 1 \), denoted as \( \Pr(x_n^{(i)} = 1) \), is known. Here we evaluate these probabilities in the adiabatic limit \( \omega_s \to 0 \).

Since \( x_n^{(1)} \) depends on \( x_n^{(2)} \) and vice versa e.g. in the case of antisymmetric response function we obtain the following system of linear equations for the above-mentioned probabilities

\[
\begin{align*}
\Pr(x_{n+1}^{(1)} = 1) &= \left[ 1 - \Pr(x_n^{(2)} = 1) \right] \Pr(x_{n+1}^{(1)} = 1|x_n^{(2)} = -1) \\
&\quad + \Pr(x_n^{(2)} = 1) \Pr(x_{n+1}^{(1)} = 1|x_n^{(2)} = 1) \\
\Pr(x_{n+1}^{(2)} = 1) &= \left[ 1 - \Pr(x_n^{(1)} = 1) \right] \Pr(x_{n+1}^{(2)} = 1|x_n^{(1)} = -1) \\
&\quad + \Pr(x_n^{(1)} = 1) \Pr(x_{n+1}^{(2)} = 1|x_n^{(1)} = 1) .
\end{align*}
\]

In the case of logistic response function in Eq. (3) \(-1\) should be replaced by 0. The conditional probabilities can be found from Eq. (1), e.g.

\[
\Pr(x_{n+1}^{(1)} = 1|x_n^{(2)} = -1) = \frac{1}{\sqrt{2\pi D}} \int_{-A \sin(\omega_s n + \phi) + w}^{\infty} e^{-\eta^2/2D^2} d\eta
\]

etc. In the adiabatic limit we assume that \( \Pr(x_n^{(i)} = 1) = \Pr(x_n^{(i)} = 1) \). Then the system (3) can be solved and the result is

\[
\begin{align*}
\Pr(x_n^{(1)} = 1) &= \frac{\alpha_n^{(1)} + (\beta_n^{(1)} - \alpha_n^{(1)}) \alpha_n^{(2)}}{1 - (\beta_n^{(1)} - \alpha_n^{(1)})(\beta_n^{(2)} - \alpha_n^{(2)})} , \\
\Pr(x_n^{(2)} = 1) &= \frac{\alpha_n^{(2)} + (\beta_n^{(2)} - \alpha_n^{(2)}) \alpha_n^{(1)}}{1 - (\beta_n^{(1)} - \alpha_n^{(1)})(\beta_n^{(2)} - \alpha_n^{(2)})} .
\end{align*}
\]

In the case of antisymmetric response function the coefficients \( \alpha_n^{(i)} , \beta_n^{(i)} \) are

\[
\begin{align*}
\alpha_n^{(1)} &= 0.5 \left\{ 1 - \text{erf}\left[ (b - A \sin(\omega_s n + \phi) + w)/\sqrt{2D} \right] \right\} , \\
\alpha_n^{(2)} &= 0.5 \left\{ 1 - \text{erf}\left[ (b - A \sin(\omega_s n + \phi + \Delta \phi) + w)/\sqrt{2D} \right] \right\} , \\
\beta_n^{(1)} &= 0.5 \left\{ 1 - \text{erf}\left[ (b - A \sin(\omega_s n + \phi) - w)/\sqrt{2D} \right] \right\} , \\
\beta_n^{(2)} &= 0.5 \left\{ 1 - \text{erf}\left[ (b - A \sin(\omega_s n + \phi + \Delta \phi) - w)/\sqrt{2D} \right] \right\} .
\end{align*}
\]

In the case of logistic response function the coefficients \( \beta_n^{(i)} \) remain the same while in the coefficients \( \alpha_n^{(i)} \) one should replace \( w \) by 0.

In order to evaluate the SNR discrete Fourier transform of the probabilities in Eq. (5) is performed

\[
P_k^{(i)} = \frac{1}{T_s} \sum_{n=0}^{T_s-1} \Pr(x_n^{(i)} = 1) \exp \left(-2\pi i nk/T_s \right) .
\]
According to Ref. 7 SNR\(^{(i)}\) normalized to a given bandwidth \(\Delta f = 1/N\) (here, \(N = 4096\)) can be evaluated from Eqs. (5) and (7) as

\[
\text{SNR}\,(i) = 10 \log \frac{N|P_1^{(i)}|^2}{\text{Pr}(x_n^{(i)} = 1) - \text{Pr}^2(x_n^{(i)} = 1)},
\]

where the bar denotes the time average over \(T_s\). The result does not depend on the initial phase \(\phi\).

The correlation function \(C\) can be also evaluated analytically. In the case of antisymmetric response function one gets

\[
\langle x_n^{(i)} \rangle = 2\text{Pr}(x_n^{(i)} = 1) - 1,
\]

\[
\langle (x_n^{(i)})^2 \rangle = 1,
\]

\[
\langle x_n^{(i)} s_n^{(i)} \rangle = 2s_n^{(i)} \text{Pr}(x_n^{(i)} = 1),
\]

while in the case of logistic response function, utilizing the fact that \(x_n^{(i)} = (x_n^{(i)})^2\) one gets

\[
\langle x_n^{(i)} \rangle = \langle (x_n^{(i)})^2 \rangle = \text{Pr}(x_n^{(i)} = 1),
\]

\[
\langle (x_n^{(i)})^2 \rangle = \text{Pr}(x_n^{(i)} = 1),
\]

\[
\langle x_n^{(i)} s_n^{(i)} \rangle = s_n^{(i)} \text{Pr}(x_n^{(i)} = 1) .
\]

After substituting the above equations in Eq. (2) together with \(\langle (s_n^{(i)})^2 \rangle = A^2/2\) one can evaluate \(C\).

Our theory, despite of simplifications, predicts the occurrence of SR in an individual element and SR with a spatiotemporal input signal for a wide range of parameters \(w\) and \(\Delta \phi\). The predictions of this theory are compared with numerical simulations in the next section. One should, however, be aware of approximations made during the derivation of the formulae for the SNR and \(C\). It is clear that the random variables \(x_n^{(1)}, x_n^{(2)}\) are mutually dependent, while in our adiabatic approximation they are treated as independent, thus Eq. (3) is only approximate. Moreover, Eq. (8) is exact only in the case of a threshold element driven by a sum of a deterministic periodic signal and white noise,\(^7\) while in Eq. (1) the total random noises \(y_n^{(i)} + w x_n^{(i)}\) are non-white because of the above-mentioned dependence between the two \(x_n^{(i)}\) variables. Thus the theoretical predictions are exact only in the limits \(w \rightarrow 0\) (uncoupled elements) and \(\omega_s \rightarrow 0\) (adiabatic limit), and the discrepancy between the numerical and analytic values of the SNR and \(C\) increases with \(|w|\).

5. Comparison Between Theoretical and Numerical Results

In this section we compare predictions based on the theory of Sec. 4 with numerical results of Sec. 3. Theory and numerical results for the ANN which consists of threshold elements with the antisymmetric response function are compared in Fig. 4, and with the logistic response function — in Fig. 5. The way of presentation is similar as in Figs. 1 and 3, respectively, and in particular the phase shifts \(\Delta \phi\) for which the pairs of figures in consecutive rows were obtained are identical as in
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Fig. 4. Comparison between numerical (symbols) and theoretical (solid curves) results for threshold elements with the antisymmetric response function: (a, c, e) SNR$^{(1)}$ versus $D$; (b, d, f) correlation function $C$ versus $D$. Phase shifts (in units of $2\pi$): (a, b) $\Delta\phi = 0$, (c, d) $\Delta\phi = 0.125$, (e, f) $\Delta\phi = 0.5$. Numbers and symbols label theoretical and numerical curves, respectively, obtained for different coupling: (1, $\bigcirc$) $w = 1.0$, (2, $\square$) $w = -0.1$, (3, $\triangle$) $w = -1.0$, (4, $+$) $w = -1.5$.

Secs. 3.1 and 3.2. In Figs. 4 and 5 numerical results for different $w$ are shown using different symbols while theoretical results for different $w$ are drawn with solid lines labeled by different numbers. From Figs. 4 and 5 one can see that certain qualitative predictions concerning the SNR and correlation function $C$ can be made using the theory of Sec. 4. In particular, the occurrence of SR with a spatiotemporal periodic signal, the disappearance of SR and, in certain cases, the enhancement of both kinds of SR due to proper coupling are predicted by this theory.

It often happens that in the limit $D \rightarrow 0$ the denominators in Eq. (5) approach zero and it was impossible to obtain reliable theoretical results in this limit even performing high-precision numerical computations. Thus the solid lines in Figs. 4 5 often start at some $D > 0$. However, the trends of theoretical curves in the limit of small noise can still be seen.
Fig. 5. Comparison between numerical (symbols) and theoretical (solid curves) results for threshold elements with the logistic (binary) response function: (a, c, e) SNR_{1} versus \( D \); (b, d, f) covariance function \( C \) versus \( D \). Phase shifts (in units of \( 2\pi \)): (a, b) \( \Delta\phi = 0.0 \), (c, d) \( \Delta\phi = 0.375 \), (e, f) \( \Delta\phi = 0.5 \). Numbers and symbols label theoretical and numerical curves, respectively, obtained for different coupling: (1, \( \circ \)) \( w = 2.0 \), (2, \( \square \)) \( w = 1.0 \), (3, \( \triangle \)) \( w = -0.1 \), (4, \( + \)) \( w = -2.0 \).

In general it can be seen that the theoretical curves \( C \) versus \( D \) agree better with the numerical ones than the curves SNR_{1} versus \( D \). This is so because when evaluating the correlation function using Eqs. (9 and 10) one does not have to make other assumptions than these made in order to obtain Eq. (5) (i.e. the validity of Eq. (3) and of the adiabatic approximation). On the contrary, the validity of Eq. (8) requires a total white random noise at the input of every element which is not the case in the system (1) as pointed out in Sec. 4. For any \( w \) the theoretical and numerical curves usually approach each other for large \( D \). In this limit the long-time correlations between the variables \( x_{n}^{(1)} \), \( x_{n}^{(2)} \) and the possibly deterministic dynamics of the system (1) are unimportant in comparison with the effect of Gaussian noises \( \eta_{n}^{(i)} \). Thus also the effect of coupling is insignificant and we approach the case of a single threshold element driven by white Gaussian noise for which the theory of Sec. 4 is exact. An overall theoretical dependence of the SNR and the correlation function on the sign of \( w \) and on \( \Delta\phi \) is also correct. In both Figs. 4 and 5 we observe...
decrease of values of the SNR and $C$ for $w > 0$ and simultaneous increase of these values for $w < 0$ as the phase shift is increased from $\Delta \phi = 0$ to $\pi$.

It is obvious that in Figs. 4 and 5 the theoretical and numerical results for small coupling, e.g. $w = -0.1$, coincide for any $\Delta \phi$. Apart from this, a more detailed analysis reveals that in the case of threshold elements with the antisymmetric response function the best agreement in the whole range of $\Delta \phi$ and $D$ is obtained for excitatory coupling, e.g. $w = 1$, and for moderate inhibitory coupling, e.g. $w = -1.0$ (Fig. 4). In the case of logistic response function the best agreement in the whole range of $\Delta \phi$ and $D$ is obtained for inhibitory coupling (Fig. 5). In the former case the disappearance of both kinds of SR for moderate inhibitory coupling and small $\Delta \phi$ is correctly predicted by the theory (Figs. 4(a)–(d)). In the latter case the enhancement of SR for large inhibitory coupling, large phase shifts, e.g. $\Delta \phi = 3\pi/4$ or $\Delta \phi = \pi$ and large $D$ is also correctly predicted (Figs. 5(c)–(f)). The best qualitative agreement between the theoretical and numerical results in the whole range of parameters $w$ and $D$ occurs for $\Delta \phi = 0$ in the case of threshold elements with the antisymmetric response function and for $\Delta \phi = \pi$ in the case of logistic response function.

The most striking discrepancy between the theory and numerical experiments is that the theoretical curves $\text{SNR}^{(i)}$ versus $D$ and $C$ versus $D$ in certain cases have a tendency to deviate from the numerical ones in the small noise limit. If $D \approx 0$ the effect of Gaussian noises $\eta_n^{(i)}$ in the system (1) becomes negligible. The influence of coupling can be strong in this case and lead to a purely deterministic dynamics of the ANN (e.g. the system remains in a fixed point or oscillates depending on the initial conditions). Such effects are neglected in the theory of Sec. 4.

For example, in the case of antisymmetric response function the theory wrongly predicts the disappearance of SR for large inhibitory coupling, cf. the curves for $w = -1.5$ in Figs. 4(c)–(f). Thus, the enhancement of SR with a spatiotemporal periodic signal by large inhibitory coupling found in Sec. 3.1 is lost in our theory. In the case of logistic response function the theory wrongly predicts the disappearance of SR e.g. for $\Delta \phi = 0$ and $\Delta \phi = 3\pi/4$ for a certain range of $w > 0$ (Figs. 5(a)–(d)), although in fact SR never disappears (Sec. 3.2). Hence, the enhancement of both SR in an individual element and SR with a spatiotemporal periodic signal e.g. for $\Delta \phi = 0$ and $w = 1$ is again lost in our theory. In both cases the respective curves have no maxima and SR disappears instead of being enhanced.

6. Summary and Conclusions

In this paper we investigated SR in the small ANN (1) consisting of two coupled threshold elements driven by independent white Gaussian noises and periodic signals shifted in phase by $\Delta \phi$. We considered SR in an individual element measured by the SNR and SR with a spatiotemporal periodic signal measured by the correlation function $C$. In the system (1) both kinds of SR occur in a wide range of coupling strengths and phase shifts, both for threshold elements with the antisymmetric and
logistic response function, although in the former case SR disappears for moderate inhibitory coupling. Moreover, we found that both kinds of SR can be enhanced due to proper coupling. This effect occurs mainly if the threshold elements have the logistic response function. Then if the phase shift $\Delta \phi$ is small there exists an optimum excitory coupling strength for which both kinds of SR are most pronounced. If the phase shift is big, e.g. $\Delta \phi > \pi/2$, SR is slightly enhanced for any inhibitory coupling. For elements with the antisymmetric response function only SR with a spatiotemporal periodic signal is enhanced for large inhibitory coupling and large phase shifts. However, in this case the largest possible values of the SNR and $C$ are usually achieved for moderate inhibitory coupling and zero noise, i.e. when both kinds of SR disappear.

So far enhancement of SR in an individual element has been reported in the cases when all elements are driven by identical periodic signals. In spatially extended systems this enhancement is connected with the maximum spatiotemporal synchronization. From our work it follows that even if there is a certain phase shift between the signals at inputs of different elements SR can still be enhanced due to proper coupling. SR is enhanced by excitory coupling if the phase shift is small and by inhibitory coupling if the phase shift is large. The latter phenomenon has not been reported so far (e.g. only diffusive coupling enhances SR in a chain of coupled bistable elements and only ferromagnetic coupling in the Ising model). This can be understood using simple arguments concerning the influence of coupling on the synchronization of firing of the two elements given in Sec. 3.1. Moreover, similar arguments can be applied to bistable systems, too. In general positive (excitory) coupling increases the probability of the two elements being in the same state at close moments of time while negative (inhibitory) coupling increases the probability that they are in the opposite state. These two states can be firing or quiescent state for threshold elements as well as the two states in the well-known bistable model of SR. Thus, using arguments similar to these in Sec. 3.1 one can expect that proper coupling of two bistable elements driven by phase-shifted signals (positive negative coupling for small and large phase shift, respectively) will also enhance SR in individual elements.

To our knowledge, processing spatiotemporal periodic signals has not been considered in the literature on SR so far. In this paper we introduced the concept of SR with a spatiotemporal periodic signal. It is defined as maximization of the correlation between the spatiotemporal input signal and the spatiotemporal system output for nonzero input noise intensity. This new application of SR appears naturally if spatially extended systems are considered. However, in order to compare this kind of SR to SR in an individual element similar measures should be used which is not the case in this paper. The SNR is a local measure of SR which can be extended to characterize SR in a spatially extended system if the power spectrum density is evaluated from a two-dimensional Fourier transform of the spatiotemporal output signal. On the other hand, SR in an individual element can also be characterized by the correlation function for one element $C(i)$. We chose two different measures
for two different kinds of SR in order to explore the possible connections between the enhancement of SR in an individual element and the spatiotemporal synchronization of the whole system to the input signal. In fact we found that both this correlation and SR in an individual element can be most pronounced for the optimum coupling strength (Sec. 3.2), but in other cases these two kinds of SR occur independently (Sec. 3.1).

In order to investigate the problems raised in this paper more thoroughly their analysis should be performed in larger spatially extended systems. The dependence of both kinds of SR on the number of elements and the spatial wavelength of the spatiotemporal periodic signal should be examined. The present results show that instead of chains of bistable stochastic elements in which array enhanced SR is usually investigated, chains of threshold elements can be used to this purpose, too.

References